

Level Two of the Quantifier Alternation Hierarchy over Infinite Words

Manfred Kufleitner and Tobias Walter

FMI, Universität Stuttgart, Germany,
 {kufleitner,walter}@fmi.uni-stuttgart.de

September 21, 2015*

The study of various decision problems for logic fragments has a long history in computer science. This paper is on the membership problem for a fragment of first-order logic over infinite words; the membership problem asks for a given language whether it is definable in some fixed fragment. The alphabetic topology was introduced as part of an effective characterization of the fragment Σ_2 over infinite words. Here, Σ_2 consists of the first-order formulas with two blocks of quantifiers, starting with an existential quantifier. Its Boolean closure is $\mathbb{B}\Sigma_2$. Our first main result is an effective characterization of the Boolean closure of the alphabetic topology, that is, given an ω -regular language L , it is decidable whether L is a Boolean combination of open sets in the alphabetic topology. This is then used for transferring Place and Zeitoun’s recent decidability result for $\mathbb{B}\Sigma_2$ from finite to infinite words.

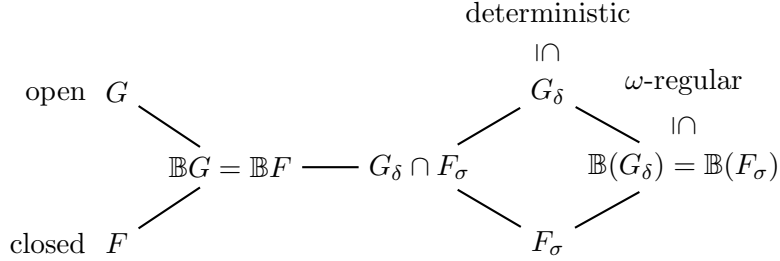
1 Introduction

Over finite words, the connection between finite monoids and regular languages is highly successful for studying logic fragments, see e.g. [3, 18]. Over infinite words, the algebraic approach uses infinite repetitions. Not every logic fragment can express whether some definable property P occurs infinitely often. For instance, the usual approach for saying that P occurs infinitely often is as follows: for every position x there is a position $y > x$ satisfying $P(y)$. Similarly, P occurs only finitely often if there is a position x such that all positions $y > x$ satisfy $\neg P(y)$. Each of these formulas requires (at least) one additional change of quantifiers, which not all fragments can provide. It turns out

*This work was supported by the German Research Foundation (DFG) under grants DI 435/5-2 and DI 435/6-1.

that topology is a very useful tool for restricting the infinite behaviour of the algebraic approach accordingly, see e.g. [4, 6, 10, 21]. In particular, the combination of algebra and topology is convenient for the study of languages in Γ^∞ , the set of finite and infinite words over the alphabet Γ . In this paper, a *regular language* is a regular subset of Γ^∞ .

Topological ideas have a long history in the study of ω -regular languages. The Cantor topology is the most famous example in this context. We write G for the Cantor-open sets and F for the closed sets. The open sets in G are the languages of the form $W\Gamma^\infty$ for $W \subseteq \Gamma^*$. If X is a class of languages, then X_δ consists of the countable intersections of languages in X and X_σ are the countable unions; moreover, we write $\mathbb{B}X$ for the Boolean closure of X . Since F contains the complements of languages in G , we have $\mathbb{B}F = \mathbb{B}G$. The Borel hierarchy is defined by iterating the operations $X \mapsto X_\delta$ and $X \mapsto X_\sigma$. The Borel hierarchy over the Cantor topology has many appearances in the context of ω -regular languages. For instance, an ω -regular language is deterministic if and only if it is in G_δ , see [8, 20]. By McNaughton's Theorem [9], every ω -regular language is in $\mathbb{B}(G_\delta) = \mathbb{B}(F_\sigma)$. The inclusion $\mathbb{B}G \subset G_\delta \cap F_\sigma$ is strict, but the ω -regular languages in $\mathbb{B}G$ and $G_\delta \cap F_\sigma$ coincide [16].



Let FO^k be the fragment of first-order logic which uses (and reuses) at most k variables. By Σ_m we denote the formulas with m quantifier blocks, starting with a block of existential quantifiers. Here, we assume that $x < y$ is the only binary predicate. We frequently identify a fragment with the languages definable therein. Let us consider FO^1 as a toy example. With only one variable, we cannot make use of the binary predicate $x < y$. Therefore, in FO^1 we can say nothing but which letters occur, that is, a language is definable in FO^1 if and only if it is a Boolean combination of languages of the form $\Gamma^*a\Gamma^\infty$ for $a \in \Gamma$. Thus $\text{FO}^1 \subseteq \mathbb{B}G$. It is an easy exercise to show that a regular language is in FO^1 if and only if it is in $\mathbb{B}G$ and its syntactic monoid is both idempotent and commutative. The algebraic condition without the topology is too powerful since this would also include the language $\{a, b\}^*a^\omega$, which is not in FO^1 . For the fragment $\mathbb{B}\Sigma_1$, the same topology $\mathbb{B}G$ with a different algebraic condition works, cf. [10, Theorems VI.3.7, VI.7.4 and VIII.4.5].

In the fragment Σ_2 , we can define the language $\{a, b\}^*ab^\omega$ which is not deterministic and hence not in G_δ . Since the next level of the Borel hierarchy already contains all regular languages, another topology is required. For this purpose, Diekert and the first author introduced the *alphabetic topology* [4]: the open sets in this topology are arbitrary unions of languages of the form uA^∞ for $u \in \Gamma^*$ and $A \subseteq \Gamma$. They showed that

a regular language is definable in Σ_2 if and only if it satisfies some particular algebraic property and if it is open in the alphabetic topology. Therefore, the canonical ingredient for an effective characterization of $\mathbb{B}\Sigma_2$ is the Boolean closure of the open sets in the alphabetic topology. Our first main result shows that, for a given regular language L , it is decidable whether L is a Boolean combination of open sets in the alphabetic topology. As a by-product, we see that every ω -regular language which is a Boolean combination of arbitrary open sets in the alphabetic topology can be written as a Boolean combination of ω -regular open sets. This resembles a similar result for the Cantor topology [16].

A major breakthrough in the theory of regular languages over finite words is due to Place and Zeitoun [13]. They showed that, for a given regular language $L \subseteq \Gamma^*$, it is decidable whether L is definable in $\mathbb{B}\Sigma_2$. This solved a longstanding open problem, see e.g. [12, Section 8] for an overview. To date, no effective characterization of $\mathbb{B}\Sigma_3$ is known. Our second main result is to show that this decidability result transfers to languages in Γ^∞ . If \mathbf{V}_2 is the algebraic counterpart of $\mathbb{B}\Sigma_2$ over finite words, then we show that \mathbf{V}_2 combined with the Boolean closure of the alphabetic topology yields a characterization of $\mathbb{B}\Sigma_2$ over Γ^∞ . Combining the decidability of \mathbf{V}_2 with our first main result, the latter characterization is effective. The proof that $\mathbb{B}\Sigma_2$ satisfies both the algebraic and the topological restrictions follows a rather straightforward approach. The main difficulty is to show the converse: every language satisfying both the algebraic and the topological conditions is definable in $\mathbb{B}\Sigma_2$.

2 Preliminaries

Words

Let Γ be a finite alphabet. By Γ^* we denote the set of finite words over Γ ; we write 1 for the empty word. The set of infinite words is Γ^ω and the set of finite and infinite words is $\Gamma^\infty = \Gamma^* \cup \Gamma^\omega$. By u, v, w we denote finite words and by α, β, γ we denote words in Γ^∞ . In this paper a *language* is a subset of Γ^∞ . Let $L \subseteq \Gamma^*$ and $K \subseteq \Gamma^\infty$. As usually L^* is the union of powers of L and $LK = \{u\alpha \mid u \in L, \alpha \in K\} \subseteq \Gamma^\infty$ is the concatenation of L and K . By L^ω we denote the set of words which are an infinite concatenation of words in L and the infinite concatenation $uu \cdots$ of the word u is written u^ω . A word $u = a_1 \dots a_n$ is a (scattered) subword of v if $v \in \Gamma^* a_1 \Gamma^* \dots a_n \Gamma^*$. The *alphabet* of a word is the set of all letters which appear in the word. The *imaginary alphabet* $\text{im}(\alpha)$ of a word $\alpha \in \Gamma^\infty$ is the set of letters which appear infinitely often in α . Let $A^{\text{im}} = \{\alpha \in \Gamma^\infty \mid \text{im}(\alpha) = A\}$ be the set of words with imaginary alphabet A . In the following we will restrict us to the study of regular languages. A language $L \subseteq \Gamma^*$ is regular if it is recognized by a (deterministic) finite automaton. A language $K \subseteq \Gamma^\omega$ is regular if it is recognized by a Büchi automaton. A language $L \subseteq \Gamma^\infty$ is regular if $L \cap \Gamma^*$ and $L \cap \Gamma^\omega$ are regular. This is equivalent to being recognized by an *extended Büchi automaton* [2].

First-Order logic

We consider first order logic FO over Γ^∞ . Variables range over the position of the word. The atomic formulas in this logic are \top for true, $x < y$ to compare two positions x and y and $\lambda(x) = a$ which is true if the word has an a at position x . One may combine those atomic formulas with the boolean connectives \neg, \wedge and \vee and quantifiers \forall and \exists . A *sentence* φ is a FO formula without free variables. We write $\alpha \models \varphi$ if $\alpha \in \Gamma^\infty$ satisfies the sentence φ . The language defined by φ is $L(\varphi) = \{\alpha \in \Gamma^\infty \mid \alpha \models \varphi\}$. We will classify the formula of FO by counting the number of quantifier alternations, that is the number of alternations of \exists and \forall . The fragment Σ_i of FO contains all FO-formula in prenex normal form with i blocks of quantifiers \exists or \forall , starting with a block of existential quantors. The fragment $\mathbb{B}\Sigma_i$ contains all Boolean combinations of formulas in Σ_i . We are particularly interested in the fragment Σ_2 and the Boolean combinations of formulas in Σ_2 . A language L is definable in a fragment \mathcal{F} (e.g. \mathcal{F} is Σ_2 or $\mathbb{B}\Sigma_2$) if there exists a formula $\varphi \in \mathcal{F}$ such that $L = L(\varphi)$, i.e., if L is definable by some $\varphi \in \mathcal{F}$. The classes of languages defined by Σ_i and $\mathbb{B}\Sigma_i$ form a hierarchy, the quantifier alternation hierarchy. This hierarchy is strict, i.e., $\Sigma_i \subsetneq \mathbb{B}\Sigma_i \subsetneq \Sigma_{i+1}$ holds for all i , cf. [1, 19].

Monomials

A *monomial* is a language of the form $A_0^* a_1 A_1^* a_2 \cdots A_{n-1}^* a_n A_n^\infty$ for $n \geq 0$, $a_i \in \Gamma$ and $A_i \subseteq \Gamma$. The number n is called the *degree*. In particular, A_0^∞ is a monomial of degree 0. A monomial is called k -monomial if it has degree at most k . In [4] it is shown that a language $L \subseteq \Gamma^\infty$ is in Σ_2 if and only if it is a finite union of monomials. We are interested in $\mathbb{B}\Sigma_2$ and thus in finite Boolean combination of monomials $A_0^* a_1 A_1^* a_2 \cdots A_{n-1}^* a_n A_n^\infty$. For this, let \equiv_k^∞ be the equivalence relation on Γ^∞ such that $\alpha \equiv_k^\infty \beta$ if α and β are contained in exactly the same k -monomials. Thus, \equiv_k^∞ -classes are Boolean combinations of monomials and every language in $\mathbb{B}\Sigma_2$ is a union of \equiv_k^∞ -classes for some k . Further, since there are only finitely many monomials of degree k , there are only finitely many \equiv_k^∞ -classes. The equivalence class of some word α in \equiv_k^∞ is denoted by $[\alpha]_k^\infty$. Note, that such a characterization of $\mathbb{B}\Sigma_2$ in terms of monomials does not yield a decidable characterization.

Our characterization of languages $L \subseteq \Gamma^\infty$ in $\mathbb{B}\Sigma_2$ is based on the characterization of languages in $\mathbb{B}\Sigma_2$ over finite words. For this, we also introduce monomials over Γ^* . A *monomial* over Γ^* is a language of the form $A_0^* a_1 A_1^* a_2 \cdots A_{n-1}^* a_n A_n^*$ for $n \geq 1$, $a_i \in \Gamma$ and $A_i \subseteq \Gamma$. The degree is defined as above. Let \equiv_k be the congruence on Γ^* which is defined by $u \equiv_k v$ if and only if u and v are contained in the same monomials over Γ^* . The equivalence classes are noted by $[u]_k$. Again, a language $L \subseteq \Gamma^*$ is in $\mathbb{B}\Sigma_2$ over Γ^* if and only if it is a union of \equiv_k -classes for some k , i.e., if $L = \cup_{u \in L} [u]_k$.

Algebra

In this paper all monoids are either finite or free. Finite monoids are a common way for defining regular languages. A monoid element e is *idempotent* if $e^2 = e$. Every element x of a finite monoid admits a unique idempotent x^i for some integer $i \geq 1$. An *ordered*

monoid (M, \leq) is a monoid equipped with a partial order which is compatible with the monoid multiplication, i.e., $s \leq t$ and $s' \leq t'$ implies $ss' \leq tt'$. Every monoid can be ordered by using the identity as partial order. For a homomorphism $h : (N, \leq) \rightarrow (M, \leq)$ between ordered monoids we require $s \leq t \Rightarrow h(s) \leq h(t)$ for all $s, t \in N$. A *divisor* is the homomorphic image of a submonoid.

A class of monoids which is closed under division and finite direct products is a *pseudovariety*. Eilenberg showed a correspondence between certain classes of languages (of finite words) and pseudovarieties [5]. A homomorphism $h : (N, \leq) \rightarrow (M, \leq)$ between two ordered monoids must hold $s \leq t \Rightarrow h(s) \leq h(t)$ for $s, t \in N$. A pseudovariety of ordered monoids is defined the same way as with unordered monoids, using the homomorphisms of ordered monoids. The Eilenberg correspondence then also holds for ordered monoids [11]. Let $\mathbf{V}_{3/2}$ be the pseudovariety of ordered monoids which corresponds to Σ_2 and \mathbf{V}_2 be the pseudovariety of monoids which corresponds to languages in $\mathbb{B}\Sigma_2$. Since $\Sigma_2 \subseteq \mathbb{B}\Sigma_2$, we obtain $\mathbf{V}_{3/2} \subseteq \mathbf{V}_2$ when ignoring the order. The connection between monoids and languages is given by the notion of *recognizability*. A language $L \subseteq \Gamma^*$ is *recognized* by an ordered monoid (M, \leq) if there is a monoid homomorphism $h : \Gamma^* \rightarrow M$ such that $L = \cup \{h^{-1}(t) \mid s \leq t \text{ for some } s \in h(L)\}$. If M is not ordered, then this means that L is an arbitrary union of languages of the form $h^{-1}(t)$.

For ω -languages $L \subseteq \Gamma^\infty$ the notion of recognizability is slightly more technical. For simplicity, we only consider recognition by non-ordered monoids. Let $h : \Gamma^* \rightarrow M$ be a monoid homomorphism. If the homomorphism h is understood, we write $[s]$ for the language $h^{-1}(s)$. We call $(s, e) \in M \times M$ a *linked pair* if $e^2 = e$ and $se = s$. By Ramsey's Theorem [14] for every word $\alpha \in \Gamma^\infty$ there exists a linked pair (s, e) such that $\alpha \in [s][e]^\omega$. A language $L \subseteq \Gamma^\infty$ is recognized by h if

$$L = \bigcup \{[s][e]^\omega \mid (s, e) \text{ is a linked pair with } [s][e]^\omega \cap L \neq \emptyset\}.$$

Since $1^\omega = 1$, the language $[1]^\omega$ also contains finite words. We thus obtain recognizability of languages of finite words as a special case. A language $L \subseteq \Gamma^\infty$ is *regular* if it is recognized by (a homomorphism to) a finite monoid.

Next, we define syntactic homomorphisms and syntactic monoids; as we will see, these are the minimal recognizers of a regular language. Let $L \subseteq \Gamma^\infty$ be a regular language. The syntactic monoid of L is defined as the quotient $\text{Synt}(L) = \Gamma^* / \approx_L$ where $u \approx_L v$ holds if and only if for all $x, y, z \in \Gamma^*$ we have both $xuyz^\omega \in L \Leftrightarrow xvyz^\omega$ and $x(uy)^\omega \in L \Leftrightarrow x(vy)^\omega \in L$. The syntactic monoid can be ordered by the quasiorder \preceq_L defined by $u \preceq_L v$ if for all $x, y, z \in \Gamma^*$ we have $xuyz^\omega \in L \Rightarrow xvyz^\omega$ and $x(uy)^\omega \in L \Rightarrow x(vy)^\omega \in L$. One can effectively compute the syntactic homomorphism of L . The syntactic monoid $\text{Synt}(L)$ satisfies the property that L is regular if and only if $\text{Synt}(L)$ is finite and the canonical homomorphism $h_L : \Gamma^* \rightarrow \text{Synt}(L)$ recognizes L , see e.g. [10, 20]. Every pseudovariety is generated by its syntactic monoids [5], i.e., every monoid in a given pseudovariety is a divisor of a direct product of syntactic monoids. The importance of the syntactic monoid of some language $L \subseteq \Gamma^\infty$ is that it is the smallest monoid recognizing L :

Lemma 1. *Let $L \subseteq \Gamma^\infty$ be a language which is recognized by a homomorphism $h : \Gamma^* \rightarrow (M, \leq)$. Then, $(\text{Synt}(L), \preceq_L)$ is a divisor of (M, \leq) .*

Proof. We assume that h is surjective and show that $\text{Synt}(L)$ is a quotient of M . If h is not surjective, we can therefore conclude that $\text{Synt}(L)$ is a divisor of M . We show that $h(u) \leq h(v) \Rightarrow u \preceq_L v$. Let u, v be words with $h(u) \leq h(v)$ and denote $h^{-1}(h(w)) = [h(w)]$ for words w . Assume $xuyz^\omega \in L$, then there exists an index i such that $(h(xuyz^i), h(z)^\omega)$ is a linked pair. Thus, $[h(xuyz^i)][h(z)]^\omega \subseteq L$ and by $h(u) \leq h(v)$ also $[h(xvyz^i)][h(z)]^\omega \subseteq L$. This implies $xvyz^\omega \in L$. The proof that $x(uy)^\omega \in L \Rightarrow x(vy)^\omega \in L$ is similar. Thus, $u \preceq_L v$ holds which shows the claim. \square

We stated the lemma for ordered monoids also for languages containing infinite words, but in the ordered setting it will be applied only for finite words.

3 Alphabetic Topology

The topological component is crucial for our approach. As mentioned in the introduction, combining algebraic and topological conditions is a successful approach for characterizations of language classes over Γ^∞ . A topology on a set X is given by a family of subsets of X (called open) which are closed under finite intersections and arbitrary unions. We define the *alphabetic topology* over Γ^∞ by its basis $\{uA^\infty \mid u \in \Gamma^*, A \subseteq \Gamma\}$. Hence, an open set is described as $\bigcup_A W_A A^\infty$ with $W_A \subseteq \Gamma^*$. The alphabetic topology has been introduced in [4], where it is used as a part of the characterization of Σ_2 over Γ^∞ .

Theorem 2 ([4]). *Let $L \subseteq \Gamma^\infty$ be a regular language. Then $L \in \Sigma_2$ if and only if $\text{Synt}(L) \in \mathbf{V}_{3/2}$ and L is open in the alphabetic topology.*

The alphabetic topology has by itself been the subject of further study [15]. We are particularly interested in Boolean combinations of open sets. An effective characterization of a language L being a Boolean combination of open sets in the alphabetic topology is given in the proposition below.

Theorem 3. *Let $L \subseteq \Gamma^\infty$ be a regular language which is recognized by $h : \Gamma^* \rightarrow M$. Then the following are equivalent:*

1. *L is a Boolean combination of open sets in the alphabetic topology where each open set is regular.*
2. *L is a Boolean combination of open sets in the alphabetic topology.*
3. *For all linked pairs $(s, e), (t, f)$ it holds that if there exists an alphabet C and words \hat{e}, \hat{f} with $h(\hat{e}) = e, h(\hat{f}) = f$, $\text{alph}(\hat{e}) = \text{alph}(\hat{f}) = C$ and $s \cdot h(C^*) = t \cdot h(C^*)$, then $[s][e]^\omega \subseteq L \Leftrightarrow [t][f]^\omega \subseteq L$.*

Proof. “1 \Rightarrow 2”: This is trivial.

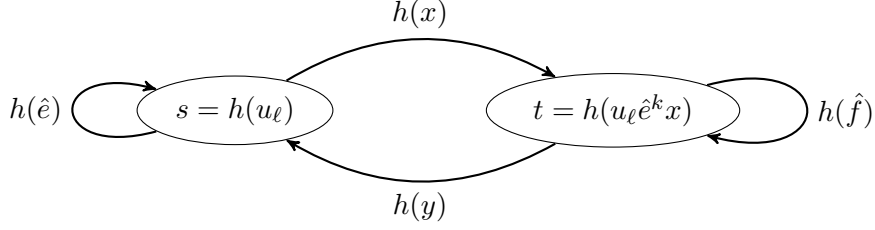


Figure 1: Part of the right Cayley graph of M in the proof of “2 \Rightarrow 3”.

“2 \Rightarrow 3”: Let L be a Boolean combination of strict alphabetic open sets. We may assume

$$L = \bigcup_{i=1}^n \left((P_i A_i^\infty) \setminus \left(\bigcup_{j=1}^{m_i} Q_{i,j} B_{i,j}^\infty \right) \right)$$

for some $P_i, Q_{i,j} \subseteq \Gamma^*$ and alphabets $A_i, B_{i,j} \subseteq \Gamma$. Assume $[s][e]^\omega \subseteq L$, but $[t][f]^\omega \not\subseteq L$. It suffices to show that $[t][f]^\omega \cap L$ is nonempty. Let $u\hat{e}^\omega \in [s][e]^\omega \subseteq L$ for some $u \in [s], \hat{e} \in [e]$ with $\text{alph}(\hat{e}) = C$. We also choose some words $\hat{f}, x, y \in C^*$ such that $h(\hat{f}) = f$, $s \cdot h(x) = t$, $t \cdot h(y) = s$ and $\text{alph}(\hat{f}) = C$.

The idea is to find an increasing sequence of words $u_\ell \in [s]$ and sets $I_\ell \subseteq \{1, \dots, n\}$ such that $u_\ell C^\infty \cap \left(P_i A_i^\infty \setminus \left(\bigcup_{j=1}^{m_i} Q_{i,j} B_{i,j}^\infty \right) \right) = \emptyset$ for all $i \in I_\ell$. We can set $u_0 = u$ and $I_0 = \emptyset$. Consider the word $u_\ell \hat{e}^\omega \in L$. There exists an index $i \in \{1, \dots, n\} \setminus I_\ell$ such that $u_\ell \hat{e}^\omega \in P_i A_i^\infty \setminus \left(\bigcup_{j=1}^{m_i} Q_{i,j} B_{i,j}^\infty \right)$. Choose k big enough, such that in the decomposition $u_\ell \hat{e}^k \hat{e}^\omega$ the part $u_\ell \hat{e}^k$ overlaps into the A_i^∞ part. Since $C = \text{alph}(\hat{e}) \subseteq A_i$, we also have $\beta_\ell = u_\ell \hat{e}^k x \hat{f}^\omega \in P_i A_i^\infty \cap A_i^{\text{im}}$. By construction we have $\beta_\ell \in [t][f]^\omega$ and therefore, assuming $[t][f]^\omega \cap L = \emptyset$, there exists an index j such that $\beta_\ell \in Q_{i,j} B_{i,j}^\infty$. Analogous, there exists a k' such that $u_\ell \hat{e}^k x \hat{f}^{k'} y C^\infty \subseteq Q_{i,j} B_{i,j}^\infty$. Hence we can choose $u_{\ell+1} = u_\ell \hat{e}^k x \hat{f}^{k'} y$ and $I_{\ell+1} = I_\ell \cup \{i\}$.

Since $u_\ell [e]^\omega \subseteq L \cap u_\ell C^\infty$, this construction has to fail at an index $\ell < n$. Therefore, the assumption is not justified and we have $[t][f]^\omega \cap L \neq \emptyset$, proving the claim.

“3 \Rightarrow 1”: Let $\alpha \in [s][e]^\omega \subseteq L$ for a linked pair (s, e) . Let $C = \text{im}(\alpha)$. By $\alpha \in [s][e]^\omega$ and the definition of C there exists an $\hat{e} \in C^*$ with $\text{alph}(\hat{e}) = C$ and $h(\hat{e}) = e$. Define

$$L' := L(s, C) := [s]C^\infty \setminus \left(\bigcup_{D \subsetneq C} \Gamma^* D^\infty \cup \bigcup_{s \notin t \cdot h(C^*)} [t]C^\infty \right).$$

We have $\alpha \in L'$ and L' is a Boolean combination of open sets in the alphabetic topology whereas each open set is regular. Since there are only finitely many sets of the type $L(s, C)$, it suffices to show $L' \subseteq L$. For $C = \emptyset$ we have $L' = [s]$ and hence $L' \subseteq L$. Thus, we may assume $C \neq \emptyset$. Let $\beta \in L'$ be an arbitrary element and let $\beta \in [t][f]^\omega$ for a linked pair (t, f) . Since β is in L' , it admits a decomposition $\beta = \tilde{v}\tilde{\beta}$ with $\tilde{v} \in [s]$

and $\tilde{\beta} \in C^\omega$. Also, by $\beta \in [t][f]^\omega$, one gets $\beta = v\beta'$ with $v \in [t], \beta' \in [f]^\omega$. Using $tf = t$ and $C \neq \emptyset$, we may assume that $|v| \geq |\tilde{v}|$, which implies $\beta' \in C^\omega$. Hence we have $t \in s \cdot h(C^*)$. By construction $\beta \notin \bigcup_{s \notin t \cdot h(C^*)} [t]C^\omega$ and therefore $s \in t \cdot h(C^*)$. It follows $s \cdot h(C^*) = t \cdot h(C^*)$. Since $\beta \notin \bigcup_{D \subsetneq C} \Gamma^* D^\omega$, we also have $\text{alph}(\beta') = C$. Using 3 it follows $\beta \in L$. \square

The alphabetic topology above is a refinement of the well-known Cantor topology. The Cantor topology is given by the basis $u\Gamma^\omega$ for $u \in \Gamma^*$. A regular language L is a Boolean combination of open sets in the Cantor topology if and only if $[s][e]^\omega \subseteq L \Leftrightarrow [t][f]^\omega \subseteq L$ for all linked pairs (s, e) and (t, f) of the syntactical monoid of L with $s \mathcal{R} t$, c.f. [4, 10, 20]. Theorem 3 is a similar result, but one had to consider the alphabetic information of the linked pairs. Hence, one does not have $s \mathcal{R} t$ as condition, but rather \mathcal{R} -equivalence within a certain alphabet C .

Remark 4. The *strict alphabetic topology* over Γ^ω , which is introduced in [4], is given by the basis $\{uA^\omega \cap A^{\text{im}} \mid u \in \Gamma^*, A \subseteq \Gamma\}$ and the open sets are of the form $\bigcup_A W_A A^\omega \cap A^{\text{im}}$ with $W_A \subseteq \Gamma^*$. Reusing the proof of Theorem 3 it turns out, that it is equivalent to be a Boolean combination of open sets in the alphabetic topology and in the strictly alphabetic topology. Since $uA^\omega = \bigcup_{B \subseteq A} uA^* B^\omega \cap B^{\text{im}}$, every open set in the alphabetic topology is also open in the strict alphabetic topology. Further, one can adapt the proof of “2 \Rightarrow 3” of Theorem 3 to show that if L is a Boolean combination of open sets in the strict alphabetic topology, then item 3 of Theorem 3 holds.

4 The fragment $\mathbb{B}\Sigma_2$

Place and Zeitoun have shown that $\mathbb{B}\Sigma_2$ is decidable over finite words. In particular, they have shown that given the syntactic homomorphism of a language L , it is decidable if $L \in \mathbb{B}\Sigma_2$. Let \mathbf{V}_2 be the pseudovariety of monoids which corresponds to the language variety of all languages contained in $\mathbb{B}\Sigma_2$. Since every pseudovariety is generated by its syntactic monoids, the result of Place and Zeitoun can be stated as follows:

Theorem 5 ([13]). *The pseudovariety \mathbf{V}_2 corresponding to the $\mathbb{B}\Sigma_2$ -definable languages in Γ^* is decidable.*

The main part of the proof will be Proposition 7. The following lemma will be an auxiliary result for Proposition 7.

Lemma 6. *There exists a number l such that for every set $\{M_1, \dots, M_d\}$ of k -monomials over Γ^* and every w with $w \in M_i$ for all $i \in \{1, \dots, d\}$, there exists a l -monomial N over Γ^* with $w \in N$ and $N \subseteq \bigcap M_i$.*

Proof. As one can iterate the statement, it suffices to show it for $d = 2$. Let $M_1 = A_0^* a_1 A_1^* a_2 \cdots A_{n-1}^* a_n A_n^*$ and $M_2 = B_0^* b_1 B_1^* b_2 \cdots B_{m-1}^* b_m B_m^*$ be two monomials. Since $w \in M_1$ and $w \in M_2$, it admits factorizations $w = u_0 a_1 u_1 a_2 \cdots u_{n-1} a_n u_n$ and $w = v_0 b_1 v_1 b_2 \cdots v_{m-1} b_m v_m$ such that $u_i \in A_i^*$ and $v_i \in B_i^*$. The factorizations mark the positions of the a_i s and the b_j s and pose an alphabetic conditions for the factors inbetween.

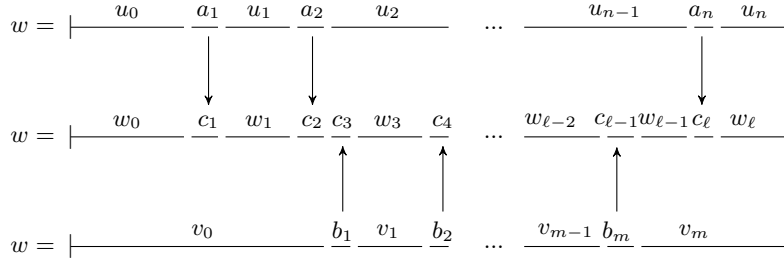


Figure 2: Different factorizations in the proof of Lemma 6. In the situation of the figure it holds $C_0 = A_0 \cap B_0$, $C_1 = A_1 \cap B_0$, $C_2 = \emptyset$, $C_3 = A_2 \cap B_1$, $C_{\ell-2} = A_{n-1} \cap B_{m-1}$, $C_{\ell-1} = A_{n-1} \cap B_m$ and $C_\ell = A_n \cap B_m$.

Thus, there exists a factorization $w = w_0 c_1 w_1 c_2 \cdots w_{\ell-1} c_\ell w_\ell$, such that the positions of c_i are exactly those, that are marked by a_i or b_j , i.e., $c_i = a_j$ or $c_i = b_j$ for some j . The words w_i are over some alphabet C_i such that $C_i = A_j \cap B_k$ for some j and k induced by the factorizations. In the case of consecutive marked positions, one can set $C_i = \emptyset$. Thus, we obtain a monomial $N = C_0^* c_1 C_1^* c_2 \cdots c_{\ell-1} C_{\ell-1}^* c_\ell C_\ell^*$ with $C_\ell = A_n \cap B_m$. By construction $N \subseteq M_1$, $N \subseteq M_2$ and $w \in N$ holds. Since there are only finitely many monomials of degree k , the size of the number l is bounded. \square

An analysis of the proof of Lemma 6 yields that the bound $l \leq n_k \cdot k$ holds, where n_k is the number of distinct k -monomials over Γ^* . Next, we will show that a language which is in \mathbf{V}_2 and is a Boolean combination of alphabetic open sets is a finite Boolean combination of monomials. One ingredient of the proof will be that by Lemma 6, we are able to compress the information of a set of k -monomials which contain a fixed word into the information that a single l -monomial contains that fixed word.

Proposition 7. *Let $L \subseteq \Gamma^\infty$ be a Boolean combination of alphabetic open sets such that $\text{Synt}(L) \in \mathbf{V}_2$. Then L is a finite Boolean combination of monomials.*

Proof. Let $h : \Gamma^* \rightarrow \text{Synt}(L)$ be the syntactic homomorphism of L and consider the languages $h^{-1}(p)$ for $p \in \text{Synt}(L)$. By Theorem 5 we obtain $h^{-1}(p) \in \mathbb{B}\Sigma_2$. Thus, there exists a number k such that for every $p \in M$ the language $h^{-1}(p)$ is saturated by \equiv_k , i.e., $u \equiv_k v \Rightarrow h(u) = h(v)$. By Lemma 6 there exists a number ℓ such that for every set $\{M_1, \dots, M_n\}$ of k -monomials and every w with $w \in M_i$ for all $i \in \{1, \dots, n\}$, there exists a ℓ -monomial N with $w \in N \subseteq \bigcap_{i=1}^n M_i$. Let $\alpha \equiv_\ell^\infty \beta$ and $\alpha \in L$. We show $\beta \in L$ which implies $L = \bigcup_{\alpha \in L} [\alpha]_\ell^\infty$ and thus that L is a finite Boolean combination of ℓ -monomials. By observing membership in $\Gamma^* C^\infty$, it is clear that $\text{im}(\alpha) = \text{im}(\beta) =: C$.

Let $u' \leq \alpha$ and $v' \leq \beta$ be prefixes such that for all every ℓ -monomial $N = N' \cdot C^\infty$ with $\alpha, \beta \in N$ we have that some prefix of u', v' is in N' . Further, let u, v be the shortest prefixes of α, β such that $u' \leq u, v' \leq v$ and for $C = \{c_1, \dots, c_m\}$ the word $(c_1 c_2 \cdots c_m)^k$ is a subword of u'' and v'' with $u = u' u''$ and $v = v' v''$, i.e., we extend the words u' and v' such that the full imaginary alphabet appears often enough. Let $\alpha = u \alpha'$ and $\beta = v \beta'$.

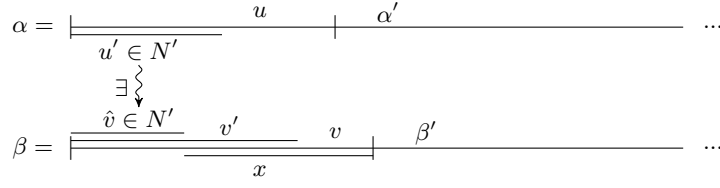


Figure 3: Factorization of α and β in the proof of Proposition 7

We use Theorem 3 and show that for $s = h(u)$ and $t = h(v)$ we have $s \cdot h(C^*) = t \cdot h(C^*)$, which implies $\beta \in L$. By symmetry, it suffices to show $t \in sh(C^*)$. Consider the set of k -monomials $N_i = N'_i C^\infty$ which hold at u , i.e., such that $u \in N'_i$ and $\alpha' \in C^\infty$. By the choice of ℓ , there exists an ℓ -monomial N' such that $u \in N'$ and $N' \subseteq \cap_i N'_i$. Since $u \in N'$, we obtain $\alpha \in N := N' C^\infty$ and by $\alpha \equiv_\ell^\infty \beta$ the membership $\beta \in N$ holds. By construction of v , there exists a prefix $\hat{v} \leq v' \leq v$ such that $\hat{v} \in N'$ and $\hat{\beta} \in C^\infty$ with $\hat{\beta}$ being defined by $\beta = \hat{v} \hat{\beta}$. Let $v = \hat{v} x$, then $x \in C^*$. We show that $ux \equiv_k v$.

Thus, let $ux \in A_0^* a_1 A_1^* a_2 \cdots A_{n-1}^* a_n A_n^*$ where the monomial has degree at most k , then there exists a factorization $A_0^* a_1 A_1^* a_2 \cdots A_{n-1}^* a_n A_n^* = M_1 M_2$ with M_1, M_2 k -monomials such that $u \in M_1$ and $x \in M_2$. By definition of N' we have $u, \hat{v} \in N' \subseteq M_1$ and thus $\hat{v} \in M_1$. We conclude that $v = \hat{v} x \in M_1 M_2 = A_0^* a_1 A_1^* a_2 \cdots A_{n-1}^* a_n A_n^*$.

Let now $v = \hat{v} x \in A_0^* a_1 A_1^* a_2 \cdots A_{n-1}^* a_n A_n^*$. Again, there exists a factorization of the monomial $A_0^* a_1 A_1^* a_2 \cdots A_{n-1}^* a_n A_n^* = M_1 M_2$ with M_1, M_2 k -monomials such that $\hat{v} \in M_1$ and $x \in M_2$. Since $(c_1 c_2 \cdots c_m)^k$ is a subword of x , there must be an A_i in M_2 such that $C \subseteq A_i$. Thus, there is a factorisation $M_2 = M_{21} M_{22}$ in k -monomials M_{21}, M_{22} such that $x' \in M_{21}$, $x'' \in M_{22}$ for $x = x' x''$ and we have $M_{21} \cdot C^* = M_{21}$. Consider $\beta = \hat{v} x \beta' \in M_1 M_{21} \cdot C^\infty$. Since $\alpha \equiv_\ell^\infty \beta$, we obtain $\alpha \in M_1 M_{21} \cdot C^\infty$. Thus, there is some prefix of u in $M_1 M_{21}$ and by $M_{21} \cdot C^* = M_{21}$, we also obtain $ux' \in M_1 M_{21}$. Thus, $ux = ux' \cdot x'' \in M_1 M_{21} \cdot M_{22} = M_1 M_2 = A_0^* a_1 A_1^* a_2 \cdots A_{n-1}^* a_n A_n^*$ holds. We conclude $ux \equiv_k v$ and thus $t = h(v) = h(u)h(x) \in sh(C^*)$. \square

The direct product of homomorphisms $g : \Gamma^* \rightarrow M$ and $h : \Gamma^* \rightarrow N$ is given by $(g \times h) : \Gamma^* \rightarrow M \times N, w \mapsto (g(w), h(w))$. It is well-known, that the direct product recognizes Boolean combinations:

Lemma 8. *Let L and K be languages such that L recognized by $g : A^* \rightarrow M$ and K is recognized by $h : A^* \rightarrow N$. Then, any Boolean combination of L and K is recognized by $(g \times h)$.*

Proof. Since $L \cap [s][e]^\omega \neq \emptyset$ implies $[s][e]^\omega \subseteq L$ for some linked pair (s, e) , we obtain $\overline{L} = \cup \{[s][e]^\omega \mid [s][e]^\omega \cap \overline{L} \neq \emptyset\}$ for the complement of L . Thus, it suffices to show that $L \cup K$ is recognized by $(g \times h)$. Obviously, L is covered by $[(s, t)][(e, f)]^\omega$, where (s, e) is a linked pair of M with $[s][e]^\omega \subseteq L$ and (t, f) is any linked pair of N . Similarly one can cover K and thus $M \times N$ recognizes $L \cup K$. \square

Next, we show that the algebraic characterisation \mathbf{V}_2 of $\mathbb{B}\Sigma_2$ over finite words also holds over finite and infinite words simultaneously. The proof of this is based on the fact that the algebraic part of the characterisation of Σ_2 over finite words and finite and infinite words is the same [4]. Since every language of Σ_2 is also a language of $\mathbb{B}\Sigma_2$, and thus $\mathbf{V}_{3/2} \subseteq \mathbf{V}_2$, combining this with Lemma 8 yields the characterization \mathbf{V}_2 .

Lemma 9. *If $L \subseteq \Gamma^\infty$ is definable in $\mathbb{B}\Sigma_2$, then $\text{Synt}(L) \in \mathbf{V}_2$.*

Proof. By definition, $L \in \mathbb{B}\Sigma_2$ implies that L is a Boolean combination of language $L_i \in \Sigma_2$. By [4] we have $\text{Synt}(L_i) \in \mathbf{V}_{3/2}$ and thus $\text{Synt}(L_i) \in \mathbf{V}_2$. Since L is a Boolean combination of L_i , L is recognized by the direct product of all $\text{Synt}(L_i)$ by Lemma 8. In particular, $\text{Synt}(L)$ is a divisor of the direct product of $\text{Synt}(L_i)$ by Lemma 1. Hence, we obtain $\text{Synt}(L) \in \mathbf{V}_2$. \square

The proof that monomials are definable in Σ_2 is straightforward.

Lemma 10. *Let $L \subseteq \Gamma^\infty$ be a monomial of the form $A_0^*a_1A_1^*a_2 \cdots A_{n-1}^*a_nA_n^\infty$. Then L is definable in Σ_2 by a formula with quantifier depth at most $n+1$.*

Proof. A formula which describes exactly the elements of the monomial is

$$\begin{aligned} \exists x_1 \dots \exists x_n \forall y : & \bigwedge_{i=1}^n \lambda(x_i) = a_i \wedge \bigwedge_{i=1}^{n-1} x_i < y < x_{i+1} \Rightarrow \lambda(y) \in A_i \wedge \\ & (y > x_n \Rightarrow \lambda(y) \in A_n) \wedge (y < x_1 \Rightarrow \lambda(y) \in A_0). \end{aligned}$$

Hence L is definable in Σ_2 . \square

Combining our results we are ready to state and prove the main theorem of the paper.

Theorem 11. *Let $L \subseteq \Gamma^\infty$ be ω -regular. Then the following are equivalent:*

1. *L is a finite Boolean combination of monomials of the form $A_0^*a_1A_1^*a_2 \cdots A_{n-1}^*a_nA_n^\infty$.*
2. *L is definable in $\mathbb{B}\Sigma_2$.*
3. *The syntactic homomorphism h of L satisfies:*

- a) *$\text{Synt}(L) \in \mathbf{V}_2$ and*
- b) *for all linked pairs $(s, e), (t, f)$ it holds that if there exists an alphabet C and words \hat{e}, \hat{f} with $h(\hat{e}) = e, h(\hat{f}) = f$, $\text{alph}(\hat{e}) = \text{alph}(\hat{f}) = C$ and $s \cdot h(C^*) = t \cdot h(C^*)$, then $[s][e]^\omega \subseteq L \Leftrightarrow [t][f]^\omega \subseteq L$.*

Proof. “1 \Rightarrow 2”: Since $\mathbb{B}\Sigma_2$ is closed under Boolean combinations, it suffices to find a formula in Σ_2 for the monomials of the form $A_0^*a_1A_1^*a_2 \cdots A_{n-1}^*a_nA_n^\infty$. Hence Lemma 10 completes the proof.

“2 \Rightarrow 3”: 3a is proved by Lemma 9. Since $A_0^*a_1A_1^*a_2 \cdots A_{n-1}^*a_n$ is a set of finite words, a monomial $A_0^*a_1A_1^*a_2 \cdots A_{n-1}^*a_nA_n^\infty$ is open in the alphabetic topology by definition. The languages in Σ_2 are unions of such monomials [4] and thus languages in $\mathbb{B}\Sigma_2$ are Boolean combinations of open sets. This implies 3b by Theorem 3.

“3 \Rightarrow 1”: This is Proposition 7. \square

$$\begin{array}{lll}
\bullet [1] = 1 & \bullet [b] = b^+ \cup (b^+ab^+)^+ & \bullet [ba] = (ba)^+ \\
\bullet [a] = a & \bullet [ab] = (ab)^+ & \bullet [aa] = \{a, b\}^*aa\{a, b\}^*
\end{array}$$
$$\varphi \equiv (\exists x \forall y: x \leq y \wedge \lambda(x) = a) \wedge (\exists x \forall y: x \geq y \wedge \lambda(x) = b) \wedge (\forall x \forall y: x \geq y \vee (\exists z: x < z < y) \vee (\lambda(x) \neq \lambda(y)))$$

12

interesting line of future work, and it may yield a decidability result for $\mathbb{B}\Sigma_2[<, +1]$ over infinite words.

Another interesting class of predicates are modular predicates. In [7] the authors have studied $\Sigma_2[<, \text{MOD}]$ over finite words. The results of [7] can be generalised to infinite words by adapting the alphabetic topology to the modular setting. As for successor predicates, we believe that an appropriate effective characterization of this topology might help in deciding $\mathbb{B}\Sigma_2[<, \text{MOD}]$ over infinite words. To the best of our knowledge however, modular predicates have not yet been considered over infinite words.

References

- [1] Janusz Antoni Brzozowski and Robert Knast. The dot-depth hierarchy of star-free languages is infinite. *J. Comput. Syst. Sci.*, 16(1):37–55, 1978.
- [2] Volker Diekert and Paul Gastin. First-order definable languages. In *Logic and Automata: History and Perspectives*, Texts in Logic and Games, pages 261–306. Amsterdam University Press, 2008.
- [3] Volker Diekert, Paul Gastin, and Manfred Kufleitner. A survey on small fragments of first-order logic over finite words. *Int. J. Found. Comput. Sci.*, 19(3):513–548, 2008.
- [4] Volker Diekert and Manfred Kufleitner. Fragments of first-order logic over infinite words. *Theory of Computing Systems*, 48(3):486–516, 2011.
- [5] Samuel Eilenberg. *Automata, Languages, and Machines*, volume B. Academic Press, 1976.
- [6] Jakub Kallas, Manfred Kufleitner, and Alexander Lauser. First-order fragments with successor over infinite words. In *STACS 2011, Proceedings*, volume 9 of *LIPIcs*, pages 356–367. Dagstuhl Publishing, 2011.
- [7] Manfred Kufleitner and Tobias Walter. One quantifier alternation in first-order logic with modular predicates. *RAIRO-Theor. Inf. Appl.*, 49(1):1–22, 2015.
- [8] Lawrence H. Landweber. Decision problems for ω -automata. *Mathematical Systems Theory*, 3(4):376–384, 1969.
- [9] Robert McNaughton. Testing and generating infinite sequences by a finite automaton. *Information and Control*, 9:521–530, 1966.
- [10] Dominique Perrin and Jean-Éric Pin. *Infinite words*, volume 141 of *Pure and Applied Mathematics*. Elsevier, 2004.
- [11] Jean-Éric Pin. A variety theorem without complementation. In *Russian Mathematics (Iz. VUZ)*, volume 39, pages 80–90, 1995.

- [12] Jean-Éric Pin. Syntactic semigroups. In *Handbook of Formal Languages*, volume 1, pages 679–746. Springer, 1997.
- [13] Thomas Place and Marc Zeitoun. Going higher in the first-order quantifier alternation hierarchy on words. In Javier Esparza, Pierre Fraigniaud, Thore Husfeldt, and Elias Koutsoupias, editors, *Automata, Languages, and Programming - 41st International Colloquium, ICALP 2014, Copenhagen, Denmark, July 8-11, 2014, Proceedings, Part II*, volume 8573 of *Lecture Notes in Computer Science*, pages 342–353. Springer, 2014.
- [14] Frank Plumpton Ramsey. On a problem of formal logic. *Proc. London Math. Soc.*, 30:264–286, 1930.
- [15] Sibylle Schwarz and Ludwig Staiger. Topologies refining the cantor topology on X^{ω} . In Cristian S. Calude and Vladimiro Sassone, editors, *Theoretical Computer Science - 6th IFIP TC 1/WG 2.2 International Conference, TCS 2010, Held as Part of WCC 2010, Brisbane, Australia, September 20-23, 2010. Proceedings*, volume 323 of *IFIP Advances in Information and Communication Technology*, pages 271–285. Springer, 2010.
- [16] Ludwig Staiger and Klaus W. Wagner. Automatentheoretische und automatenfreie Charakterisierungen topologischer Klassen regulärer Folgenmengen. *Elektron. Inform.-verarb. Kybernetik*, 10:379–392, 1974.
- [17] Howard Straubing. Finite semigroup varieties of the form $\mathbf{V} * \mathbf{D}$. *Journal of Pure and Applied Algebra*, 36(1):53–94, 1985.
- [18] Howard Straubing. *Finite Automata, Formal Logic, and Circuit Complexity*. Birkhäuser, 1994.
- [19] Wolfgang Thomas. Classifying regular events in symbolic logic. *J. Comput. Syst. Sci.*, 25:360–376, 1982.
- [20] Wolfgang Thomas. Automata on infinite objects. In *Handbook of Theoretical Computer Science*, chapter 4, pages 133–191. Elsevier, 1990.
- [21] Thomas Wilke. Locally threshold testable languages of infinite words. In *STACS '93, Proceedings*, volume 665 of *LNCS*, pages 607–616. Springer, 1993.